

Brane Kantowski-Sachs Universe with Linear Equation of State and Future Singularity

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We consider the brane Kantowski-Sachs universe when bulk space is five dimensional anti-de Sitter obeying a linear equation of state of the form $p = (\gamma - 1)\rho + p_0$, where γ, p_0 are parameters. In this framework we discuss the brane Kantowski-Sachs cosmology and compare it with standard Kantowski-Sachs cosmology in Einstein theory for some specific matter choice. For several specific choice of relation between energy and pressure, the behavior of the scale factors at early time is found. It is shown that the brane Kantowski-Sachs cosmology may quickly become isotropic for some choices of matter. We also discuss the universe fate of Kantowski-Sachs cosmological model by means of the rip or singularity in both the theories.

I. INTRODUCTION

The brane world scenario assumes that our universe is a four dimensional space-time, a 3-brane, embedded in the five dimensional (5D) bulk space-time. Moreover inspired by the string theory/M-theory, the Z_2 -symmetry with respect to the brane is imposed [1]. Randall and Sundrum [2]–[3] have shown that a scenario with an infinite fifth dimension in the presence of brane can generate the theory of gravity which mimics purely four dimensional gravity, both with respect to the classical gravitational potential and with respect to gravitational relation.

In the standard as well as in the brane cosmology it is quite possible that early universe could be the anisotropic one. The behavior of an anisotropic Bianchi type I brane-world in the presence of inflationary scalar fields has been considered by Maartens et al. [4]. Also Obukhov et al. [5] considered the brane Kantowski-Sachs (KS) universe and compared the KS brane cosmology with KS cosmology for Einstein theory for several classical matter choices obeying the barotropic equation of state of the form $p = (\gamma - 1)\rho$.

The cosmological observation indicate that the expansion of our universe accelerated [[7]–[8]]. In the last decade lots of work on its possible mechanism, such as extended gravity [9], modifying equation of state are to explain the cosmic acceleration expansion observed. To over-come the drawback of hydrodynamics instability, Babichev et al. [10] was proposed a linear equation of state of more general from $p = \alpha(\rho - \rho_0)$ and this form incorporated into cosmological model can describe the hydrodynamical stable dark energy behaviors. Meng et al. [9] solved the Friedmann equations with both more general equation of state of the form

$$p = (\gamma - 1)\rho + p_0, \quad 1 \leq \gamma \leq 2, \quad (1)$$

where γ, p_0 are parameters, and the bulk viscosity, and discussed the acceleration expansion of the universe evolution and future singularity for the model.

In this paper we generalize the work of Obukhov et al. [5] by assuming more general equation of state (EOS) of the form Eq.(1) and compare the brane Kantowski-Sachs (KS) cosmology with KS cosmology for Einstein theory for several specific choices of relation between energy and pressure and also discuss the fate of universe evolution by means of the rip or singularity.

II. BRANE WORLD AND KANTOWSKI-SACHS GEOMETRY

The brane world scenario was proposed by Randall and Sandrum [3]. They have shown that the in a five dimensional space-time (called bulk) it is possible to confine the matter field in four dimensional hypersurface called (3-brane). The effective action in bulk is [11]

$$S = \int d^5 \sqrt{-g_5} \left(\frac{1}{2k_5^2} R_5 - \Lambda_5 \right) + \int_{\chi=0} d^4 \sqrt{-g} \left(\frac{1}{2k_5^2} K^\pm + \lambda + L^{matter} \right). \quad (2)$$

Here any quantity in the bulk is marked with subscript '5' with $k_5^2 = 8\pi G_5$ being the five dimensional gravitational coupling constant. The coordinate on the bulk are denoted by x^ν , ($\nu = 0, 1, 2, 3, 4$) while those on the brane as the x^μ , ($\mu = 0, 1, 2, 3$). R_5 is the 5D intrinsic curvature in the bulk and K^\pm is the intrinsic curvature on either side of the brane.

On the 5D space-time (the bulk), with the negative vacuum energy Λ_5 as the only source of the gravitational field the Einstein field equations are given by

$${}^{(5)}G_{IJ} = k_5^2,$$

$${}^{(5)}T_{IJ} = -\Lambda_5 + {}^{(5)}g_{IJ} + \delta(\chi)[- \lambda g_{IJ} + T_{IJ}^{matter}]. \quad (3)$$

In this space-time a brane is a fixed point of Z_2 symmetry.

Assuming a metric of the form $ds^2 = (n_I n_J + g_{IJ})dx^I dx^J$, with $n_I dx^I = d\chi$ the unit normal to the

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χ = constant hypersurface and g_{IJ} the induced metric on χ = constant hypersurfaces, the effective four dimensional gravitational equations on the brane takes the form [12]–[14]

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + k_4^2 T_{\mu\nu} + k_5^4 S_{\mu\nu} - E_{\mu\nu}, \quad (4)$$

where $G_{\mu\nu}$ and $T_{\mu\nu}$ are the Einstein and energy momentum tensors and $S_{\mu\nu}$ is the quadratic contribution on $T_{\mu\nu}$ defined as

$$S_{\mu\nu} = \frac{1}{12} T T_{\mu\nu} - \frac{1}{4} T_\mu^\alpha T_{\alpha\nu} + \frac{1}{24} g_{\mu\nu} (3 T^{\alpha\beta} T_{\alpha\beta} - T^2),$$

and $\Lambda = k_5^2(\Lambda_5 + k_5^2\lambda^2/6)$, $k_4^2 = k_5^2\lambda^2/6$ and $E_{IJ} = C_{IAJB} n^A n^B$. C_{IAJB} is the five dimensional Weyl tensor in the bulk and λ is the vacuum energy on the brane. $T_{\mu\nu}$ is the matter energy momentum tensor on the brane with the components $T_0^0 = -\rho$, $T_1^1 = T_2^2 = T_3^3 = p$ and $T = T_\mu^\mu$ is the trace of the energy momentum tensor. In this paper we restrict our analysis to a flat bulk geometry with $C_{IAJB} = 0$.

We will consider the brane metric in Kantowski-Sachs from [15]–[16]

$$ds^2 = -dt^2 + R_1^2(t) dr^2 + R_2^2(t) (d\theta^2 + \sin^2\theta d\phi^2). \quad (5)$$

For later convenience, we will introduce the following variables

$$V = R^3 = R_1 R_2^2, \quad A_m = \frac{\sum_{i=1}^3 (H_i - H)^2}{3H^2}, \quad (6)$$

$$H_i = \frac{\dot{R}_i}{R_i}, \quad H = \frac{1}{3}(H_1 + 2H_2) = \frac{\dot{V}}{3V}, \quad q = \frac{d}{dt}(H^{-1}) - 1, \quad (7)$$

where R_1, R_2 are scale factors, R average scale factor, V is the volume scale factor, H_1, H_2 are Hubble factors, H is average Hubble factor, q is deceleration parameter, and A_m anisotropy parameter.

III. CONVENTIONAL EINSTEIN THEORY WITH LINEAR EQUATION OF STATE

For the conventional Einstein theory (CET) the Einstein field equations and energy conservation law

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + k_4^2 T_{\mu\nu}, \quad \nabla_\mu T^{\mu\nu} = 0, \quad (8)$$

with $G_{\mu\nu}$ being the Einstein tensor (4D), Λ the cosmological constant and k_4 the gravitational coupling, $k_4^2 = 8\pi G$.

Einstein field equations and conservation law Eq.(8) for the Kantowski-Sachs universe Eq. (5) with linear equation of state (LEOS) of the form Eq. (1) becomes

$$\frac{\dot{R}_2^2}{R_2^2} + 2 \frac{\dot{R}_1 \dot{R}_2}{R_1 R_2} + \frac{1}{R_2^2} = \Lambda + k_4^2 \rho, \quad (9)$$

$$\frac{2\ddot{R}_2}{R_2} + \frac{\dot{R}_2^2}{R_2^2} + \frac{1}{R_2^2} = \Lambda + k_4^2 \rho (1 - \gamma) - k_4^2 p_0, \quad (10)$$

$$\frac{2\ddot{R}_1}{R_1} + \frac{\ddot{R}_2}{R_2} + \frac{\dot{R}_1 \dot{R}_2}{R_1 R_2} = \Lambda + k_4^2 \rho (1 - \gamma) - k_4^2 p_0, \quad (11)$$

$$\dot{\rho} + (\gamma\rho + p_0) \left(\frac{\dot{R}_1}{R_1} + 2 \frac{\dot{R}_2}{R_2} \right) = 0. \quad (12)$$

First of all, Eq. (12) can be easily solved to obtain the time evolution law of the energy density of the fluid

$$\rho = \rho_0 V^{-\gamma} - \gamma^{-1} p_0, \quad \rho_0 = \text{constant} > 0. \quad (13)$$

We can write the above field equations in the another form

$$\frac{d}{dt}(V H_1) = \left[\Lambda - \frac{k_4^2 p_0}{\gamma} \right] V + \frac{1}{2} k_4^2 V^{(1-\gamma)} \rho_0 (2 - \gamma), \quad (14)$$

$$\frac{d}{dt}(V H_2) = \left[\Lambda - \frac{k_4^2 p_0}{\gamma} \right] V + \frac{1}{2} k_4^2 V^{(1-\gamma)} \rho_0 (2 - \gamma) - R_1, \quad (15)$$

$$3\dot{H} + H_1^2 + 2H_2^2 = \left[\Lambda - \frac{3k_4^2 p_0}{2\gamma} \right] + \frac{1}{2} k_4^2 V^{-\gamma} \rho_0 (2 - 3\gamma). \quad (16)$$

From Eq. (14) and Eq. (15), the equation for V can be written as

$$\ddot{V} = 3 \left[\Lambda - \frac{k_4^2 p_0}{\gamma} \right] V + \frac{3}{2} k_4^2 V^{(1-\gamma)} \rho_0 (2 - \gamma) - 2R_1. \quad (17)$$

After first integration

$$\dot{V}^2 = 3 \left[\Lambda - \frac{k_4^2 p_0}{\gamma} \right] V^2 + 3k_4^2 V^{(2-\gamma)} \rho_0 - 4 \int R_1 dV. \quad (18)$$

Again from Eq. (14) and Eq. (15)

$$\frac{d}{dt}[V(H_1 - H_2)] = -R_1. \quad (19)$$

It follows that

$$H_1 = H + \frac{2}{3V} K, \quad H_2 = H - \frac{1}{3V} K, \quad K = \int R_1 dt. \quad (20)$$

IV. ASYMPTOTIC BEHAVIOR IN CET

In this section we consider the asymptotic behavior for V in CET.

A. For large V

If V is large then RHS of Eq. (18) $\rightarrow 3 \left[\Lambda - \frac{k_4^2 p_0}{\gamma} \right] V^2$, when $\Lambda - \frac{k_4^2 p_0}{\gamma} > 0$. This leads to the volume scale factor of our universe to expand, exponentially in the large time limit i.e. $V \propto \exp \left[\sqrt{3 \left(\Lambda - \frac{k_4^2 p_0}{\gamma} \right)} t \right]$ and $H_1 = H_2$.

By using Eq. (6) and (7) we get

$$R_1 = \frac{a}{b^2} \exp[\sqrt{(\Lambda_0/3)}t], \quad R_2 = b \exp[\sqrt{(\Lambda_0/3)}t], \quad (21)$$

where a, b are constants and $\Lambda_0 = (\Lambda - k_4^2 p_0 \gamma^{-1})$.

Therefore, in this limit the mean anisotropic parameter A_m decays to zero exponentially and the deceleration parameter q becomes negative i.e. $q < 0$. Hence the universe can be isotropized dynamically and undergoes an accelerated expansion in the large time limit due to the presence of the term $\Lambda_0 > 0$ when $p_0 < 0$. In fact, the value of Λ_0 can only change the expanding rate of the universe but not affect its effect of isotropization in general. The behavior of the system does not depend on γ and the anisotropic CET universe becomes isotropic one for large V due to the classical matter effects.

B. For extremely small V

In this case the properties of the CET universe will depend on the value of γ . When V is extremely small at the limit $t \rightarrow 0$, the RHS of Eq. (18) $\rightarrow 3k_4^2 \rho_0 V^{(2-\gamma)} + b$, where b is constant. Therefore, the mean anisotropy parameter A_m at the early universe tends to zero.

Hence from Eq. (18)

$$\dot{V} = \sqrt{6a\rho_0 V^{(2-\gamma)} + b}, \quad (22)$$

where $a = \frac{1}{2}k_4^2 \rho_0$.

For $\gamma = 3/2$, the solution takes the form

$$V = \frac{9}{4}a^{3/2}(t-t_0)^{4/3}, \quad R_1 = \frac{9}{4c^2}a^{3/2}(t-t_0)^{4/9},$$

$$R_2 = c(t-t_0)^{4/9}, \quad (23)$$

where c is the constant of integration and also supposed that $V(t_0) = 0$.

For $\gamma = 1$, the solutions takes the form

$$V \propto (t-t_0)^2, \quad R_1 \propto (t-t_0)^{2/3}, \quad R_2 \propto (t-t_0)^{2/3}. \quad (24)$$

In this case the solution is also isotropic. Hence it is observed that for the large and the sufficiently small V the solution becomes isotropic. It is note that even for small cosmological time the process of isotroization starts quickly [5].

V. BRANE KANTOWSKI-SACHS UNIVERSE WITH LINEAR EQUATION OF STATE

In this section we consider the brane cosmology with zero Weyl tensor i.e. $C_{IAJB}n^A n^B = 0$.

The gravitational field equations (4) and the conservation law on the brane with LEOS (1) takes the form

$$\frac{\dot{R}_2^2}{R_2^2} + 2\frac{\dot{R}_1 \dot{R}_2}{R_1 R_2} + \frac{1}{R_2^2} = \Lambda + k_4^2 \rho + \frac{1}{12}k_5^4 \rho^2, \quad (25)$$

$$\frac{2\ddot{R}_2}{R_2} + \frac{\dot{R}_2^2}{R_2^2} + \frac{1}{R_2^2} = (\Lambda - k_4^2 p_0) - k_4^2 \gamma \rho + \frac{1}{12}k_5^4 \rho^2 (1 - 2\gamma), \quad (26)$$

$$\frac{2\ddot{R}_1}{R_1} + \frac{\dot{R}_2}{R_2} + \frac{\dot{R}_1 \dot{R}_2}{R_1 R_2} = (\Lambda - k_4^2 p_0) - k_4^2 \gamma \rho + \frac{1}{12}k_5^4 \rho^2 (1 - 2\gamma), \quad (27)$$

$$\dot{\rho} + (\gamma \rho + p_0) \left(\frac{\dot{R}_1}{R_1} + 2\frac{\dot{R}_2}{R_2} \right) = 0. \quad (28)$$

Here we consider $\left(\frac{k_5^4 p_0}{6} \right) = k_4^2$ and p_0 plays the role of λ , the vacuum energy on the brane.

From Eq. (28) after integrating with the help of Eqs. (6) and (7) we get

$$\rho = \rho_0 V^{-\gamma} - \gamma^{-1} p_0, \quad \rho_0 = \text{constant} > 0. \quad (29)$$

The above field equations in another form as

$$\frac{d}{dt}(V H_1) = \left[\Lambda - \frac{k_4^2 p_0 (2\gamma - 1)}{2\gamma^2} \right] V$$

$$+ \frac{1}{2\gamma} k_4^2 V^{(1-\gamma)} \rho_0 (1-\gamma)(\gamma-2) + \frac{k_5^4}{12} \rho_0^2 (1-\gamma) V^{(1-2\gamma)}, \quad (30)$$

$$\frac{d}{dt}(V H_2) = \left[\Lambda - \frac{k_4^2 p_0 (2\gamma - 1)}{2\gamma^2} \right] V$$

$$+ \frac{1}{2\gamma} k_4^2 V^{(1-\gamma)} \rho_0 (1-\gamma)(\gamma-2) + \frac{k_5^4}{12} \rho_0^2 (1-\gamma) V^{(1-2\gamma)} - R_1, \quad (31)$$

$$3\dot{H} + H_1^2 + 2H_2^2 = \left[\Lambda - \frac{k_4^2 p_0 (2\gamma^2 + 2\gamma - 1)}{2\gamma^2} \right]$$

$$+ k_4^2 V^{-\gamma} \rho_0 (1+3\gamma)(1-3\gamma) + \frac{1}{12} k_4^2 V^{-2\gamma} \rho_0^2 (1-3\gamma). \quad (32)$$

From Eqs. (30) and (31) the equation for V can be obtained as

$$\ddot{V} = 3 \left[\Lambda - \frac{k_4^2 p_0 (2\gamma^2 + 2\gamma - 1)}{2\gamma^2} \right] V$$

$$+\frac{3}{2\gamma}k_4^2V^{(1-\gamma)}\rho_0(1-\gamma)(\gamma-2)-2R_1+\frac{1}{4}k_4^2V^{(1-2\gamma)}\rho_0^2(1-\gamma). \quad (33)$$

After first integration we get

$$\dot{V} = \sqrt{F(V)}, \quad (34)$$

where

$$F(V) = 3 \left[\Lambda - \frac{k_4^2 p_0 (2\gamma - 1)}{2\gamma^2} \right] V^2 - 3k_4^2 V^{(2-\gamma)} \rho_0 \left(\frac{1-\gamma}{\gamma} \right) + \frac{1}{4}k_4^2 V^{2(1-\gamma)} \rho_0^2 - 4 \int R_1 dV. \quad (35)$$

Again from Eqs. (30) and (31)

$$H_1 = H + \frac{2}{3V}K, \quad H_2 = H - \frac{1}{3V}K, \quad K = \int R_1 dt. \quad (36)$$

VI. ASYMPTOTIC BEHAVIOR IN BRANE COSMOLOGY

In the following part we will discuss the asymptotic behavior of for large and small V .

In the brane cosmology the asymptotic behavior for the large V is similar to the CET. However, for the extremely small V one obtains difference in comparison with CET.

A. For extremely small V

For extremely small V and $\gamma = 1$ the RHS of Eq. (34) $\rightarrow (a^2 + b)$, where $a^2 = \left(\frac{k_5^4 \rho_0^2}{4} \right)$ and b is constant.

Then from Eq. (34)

$$V = C_1 + \sqrt{a^2 + b} t, \quad (37)$$

where C_1 is the constant of integration.

From Eq. (36)

$$R_1 = C_1 V^{1/3} \exp \left[\frac{2K}{3} \int V^{-1} F(V)^{-1/2} dV \right]. \quad (38)$$

After substituting the value of V from Eq. (37) into Eq. (38) we get

$$R_1 = C_2 [C_1 + \sqrt{a^2 + b} t]^{\frac{1}{3} + \frac{2K}{3\sqrt{a^2 + b}}}, \quad (39)$$

$$R_2 = \frac{1}{\sqrt{C_2}} [C_1 + \sqrt{a^2 + b} t]^{\frac{1}{3} - \frac{K}{3\sqrt{a^2 + b}}}. \quad (40)$$

For $\gamma = 3/2$, we get

$$R_1 = \frac{1}{C_2^2} \left(\frac{3a}{2} \right)^{2/3} (t - t_0)^{2/9}, \quad (41)$$

$$R_2 = C_2 (t - t_0)^{2/9}, \quad V = \left(\frac{3a}{2} \right)^{2/3} (t - t_0)^{2/3}. \quad (42)$$

Here C_1, C_2 are constants.

VII. CONCLUSIONS AND DISCUSSIONS

In the present work we have compared the KS cosmological model in CET and the brane theory by using LEOS. The two different theories give, however completely different anisotropic expansion at the very early stages of evolution. From Eq. (4), it is easy to realize that the brane world scenario is different from the CET by two parts: (i) the matter fields contribute local "quadratic" energy momentum tensor correction via the tensor $S_{\mu\nu}$, and (ii) there are "nonlocal" effects from the free field in the bulk, transmitted via projection of the bulk Weyl tensor $E_{\mu\nu}$. Therefore, the CET can be treated as a limit of the brane theory by taking $E_{\mu\nu} = 0$ and $k_5 \rightarrow 0$ with properly adjusted values of the constant k_4 and Λ [17]. The CET can be reproduced by imposing the limit $k_5 \rightarrow 0$.

Indeed, for the CET the anisotropic expansion tends to be large at the very early stage. In the another words, the universe has to begin from highly anisotropic initial expansion and then decay to zero as the time increase. In the other hand, in the brane cosmology, due to the quadratic correction which sufficiently changes the early time behavior of the universe.

It is observed that for $\gamma = 3/2$, brane KS universe become isotropic at early time analogous to CET case. The nature of the scale factors for $\gamma = 3/2$, in both the cases are slightly different, which shows the role of five dimensional bulk. For $\gamma = 1$, KS cosmology for CET become isotropic and brane KS universe remains anisotropic. It is observed that in both the model we don't get the exact solution. However, when $R_1 = \text{constant}$, $\Lambda = 0$, $p_0 = 0$ and $\gamma = 2$, we get the exact solution only in case of CET. Therefore, the equation of state most appropriate to describe the high density regime of the early universe is the stiff Zeldovich one, with $\gamma = 2$. [18].

It is worth noting that for $p = p_0$ i. e. $\gamma = 1$, mean anisotropy of the KS brane model behave similar to the models CET where the mean anisotropy parameter A_m is large in the very early time.

We are able to investigate the fate of our cosmological models by means of the rip or singularity. The singularity behavior is classified in [19] - [20]:

- Type I (Big Rip): For $t \rightarrow t_s$, $a \rightarrow \infty$, $\rho \rightarrow \infty$, and $|p| \rightarrow \infty$,
- Type II (Sudden Singularity): For $t \rightarrow t_s$, $a \rightarrow a_s$, $\rho \rightarrow \rho_s$, and $|p| \rightarrow \infty$,
- Type III: For $t \rightarrow t_s$, $a \rightarrow a_s$, $\rho \rightarrow \infty$ and $|p| \rightarrow \infty$,
- Type IV: For $t \rightarrow t_s$, $a \rightarrow a_s$, $\rho \rightarrow 0$, and $|p| \rightarrow 0$ and p and ρ are finite and higher order derivatives of H diverge.

For CET with $\gamma = 3/2$ and $\gamma = 1$ at $t \rightarrow t_0$, the average scale factor $R \rightarrow 0$ and $\rho \rightarrow \infty$, so the type III singularity occurs. Also for the KS brane cosmology with $\gamma = 3/2$, there will be again a type III singularity. However, for

$\gamma = 1$ in KS brane cosmology there is no future singular-

ity. This shows the qualitative difference between CET and brane cosmology.

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- [1] P. Horava E. Witten, *Nucl.Phys.* **B460** 506 (1996).
 - [2] L. Randall and R Sundrum, *Phys Rev. Lett.* **83** 3370 (1999).
 - [3] L. Randall and R. Sundrum, *Phys. Rev. Lett.* **83** 4690 (1999).
 - [4] R. Maartens, V. Sahni, and T. D. Saini, *Phys. Rev. D* **63** 063509 (2001).
 - [5] V. V. Obukhov, A. V. Makarenko and K. E. Osetrin, *J. Phys A: Math. Gen.* **39** 6635 (2006).
 - [6] S. Perlmutter et al., *Nature (London)* **391** 51 (1998).
 - [7] T. Totani, Y. Yoshii and K. Sato, *Astrophys J.* **483** L75 (1999) .
 - [8] A. G. Riess et al., *Astrophys J.* **116** 1009 (1998).
 - [9] X. H. Meng, J. Ren and M. G. Hu, *Commun. Theor. Phys.* **47** 379 (2007).
 - [10] E. Babichev, V. Dokuchaev and Y. Eroshenko, *Class. Quantum Grav.* **22** 13 (2005).
 - [11] K. Maeda and D. Wands, *Phys. Rev. D* **62** 124009 (2000).
 - [12] M. Sasaki, T. Shiromizu and K. Maeda, *Phys. Rev. D* **62** 024008 (2000).
 - [13] T. Shiromizu, K. Maeda and M. Sasaki, *Phys. Rev. D* **62** 024012 (2000).
 - [14] R. Maartens, *Phys. Rev. D* **62** 084023 (2000).
 - [15] S. Nojiri, O. Obergon, S. D. Odintos and K. E. Osetain, *Phys. Rev. D* **60** 024008 (1999).
 - [16] R. Kantowski and R. Sachs, *J. Math. Phys* **7** 443 (1966).
 - [17] C. M. Chen and W. F. Kao, *hep-th/0201188* (2002).
 - [18] C. M. Chen, T. Harko and M. K. Mak, *Phys. Rev. D* **64** (044013) (2001).
 - [19] S. Nojiri, S. D. Odintsov and S. Tsujikawa, *Phys. Rev. D* **71** (063004) (2005).
 - [20] G. S. Khadekar and N. V. Gharad, *The Open Astronomy Journal* **7** 1 (2014).